Measurable Brooks's Theorem for Directed Graphs

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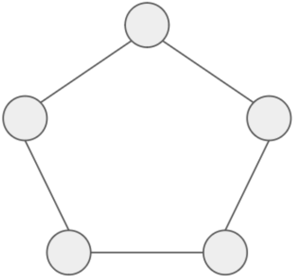
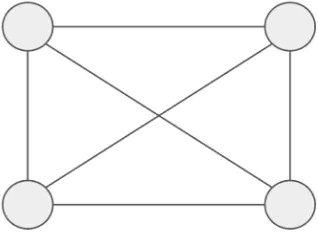


Odd cycles



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Fact

[Kechris–Solecki–Todorčević, 1999] There is a Borel graph G for which $\chi(G) = 2$ but $\chi_B(G)$ is uncountable.

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In fact, the set of acyclic d-regular Borel graphs having Borel chromatic number at most d is $\mathbf{\Sigma}^1_2$-complete [Brandt–Chang–Grebík–Grunau–Rozhoň–Vidnyánszky, 2024].

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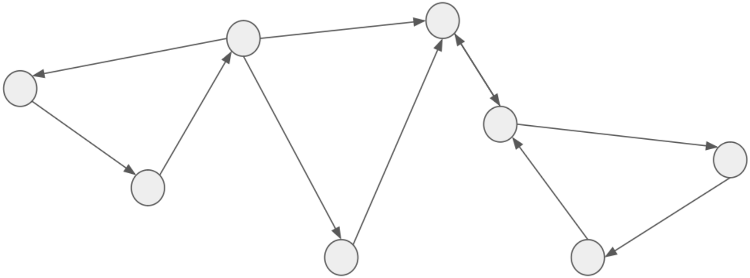

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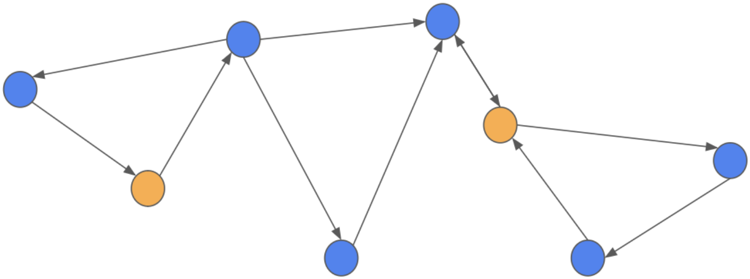

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[Harutyunyan–Mohar, 2011] Suppose the maximum degree of each vertex in D is at most $d \geq 2$. If $d = 2$, suppose D has no symmetrizations of odd cycles; if $d \geq 3$, suppose D does not contain the symmetrization of the complete graph on $d + 1$ vertices. Then $\overrightarrow{\chi}(D) \leq d$.

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The directed Schreier graph of this action with generators $\gamma_0, \gamma_1, \gamma_2$ has no Borel 3-dicoloring.

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3. Dicolor the "hard" components, the infinite Gallai trees, *by discarding a μ-null set*.

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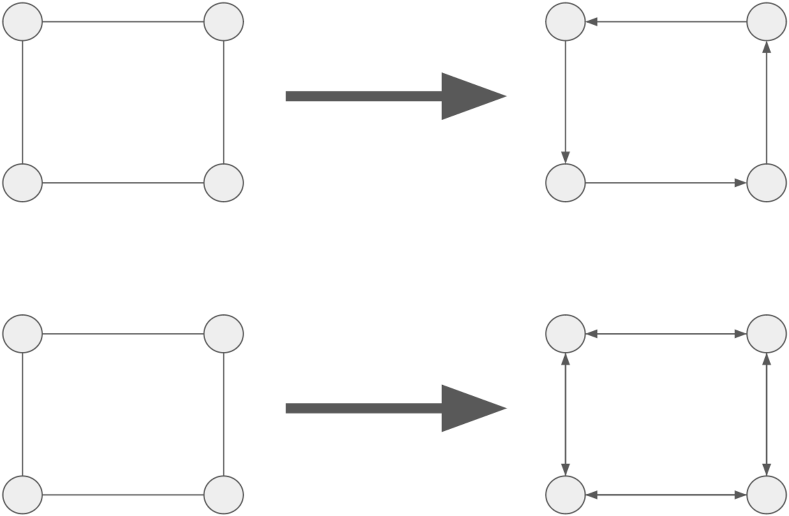
- A connected digraph is a **Gallai tree** if all its blocks are bad.

Gallai Trees

Underlying graph

Directed graph



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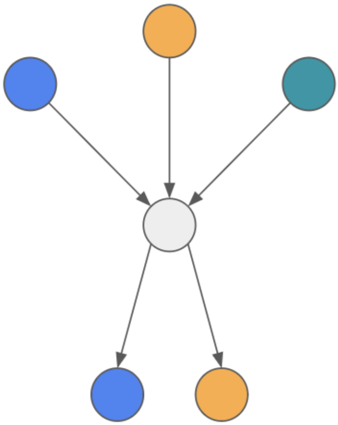

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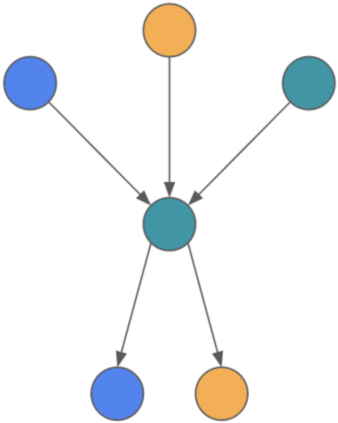

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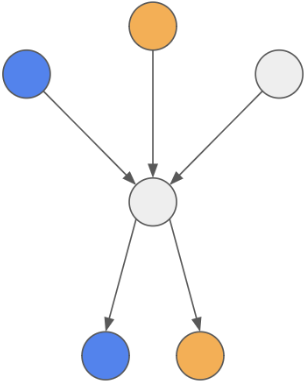

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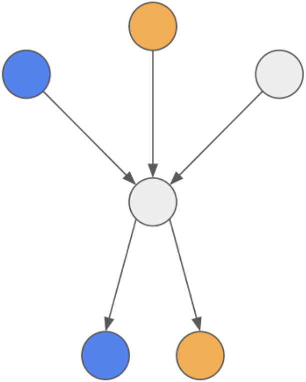

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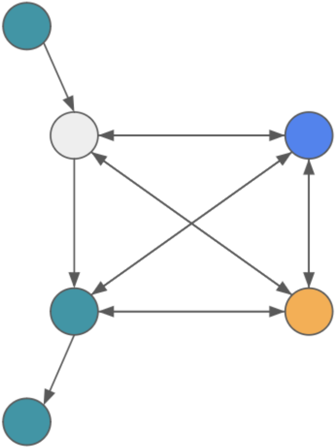

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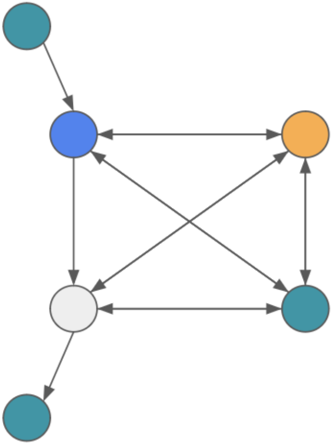

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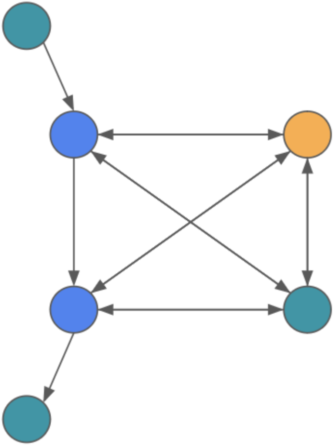

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Select the reserved good blocks so that, for distinct selected blocks S, T, no neighbor of S is also a neighbor of T.

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It may be instructive to consider the μ-measurable or Baire-measurable setting.

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2. What can we say about LOCAL algorithms for dicoloring? Are there implications between descriptive digraph combinatorics and LOCAL dicoloring algorithms?

Thank you!